SOME EVIDENCE IN FAVOR OF MORREY'S CONJECTURE

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Abstract. We provide further evidence to favor the fact that rank-one convexity does not imply quasiconvexity for two-component maps in dimension two. We provide an explicit family of maps parametrized by τ, and argue that, for small τ, they cannot be achievable by lamination. In this way, Morrey's conjecture might turn out to be correct in all cases.

1. Introduction

One of the main ingredients of the direct method of the Calculus of Variations ([10]) to show existence of minimizers for an integral functional of the kind

$$I(\mathbf{u}) = \int_\Omega \psi(\nabla \mathbf{u}(\mathbf{x}))\, d\mathbf{x}$$

is its weak lower semicontinuity. Here $\Omega \subset \mathbf{R}^N$ is a regular, bounded domain, and feasible mappings $\mathbf{u} : \Omega \to \mathbf{R}^m$ are smooth or Lipschitz, so that $\nabla \mathbf{u}$ is a $m \times N$-matrix at each point $\mathbf{x} \in \Omega$. The weak lower semicontinuity property is in turn equivalent to suitable convexity properties of the continuous integrand $\psi : \mathbf{M}^{m\times N} \to \mathbf{R}$. Morrey ([21], [22]) proved that this weak lower semicontinuity (in $W^{1,\infty}(\Omega; \mathbf{R}^m)$) is equivalent to the quasiconvexity of the integrand ψ, namely,

$$\psi(\mathbf{F}) \le \frac{1}{|D|} \int_D \psi(\mathbf{F} + \nabla \mathbf{v}(\mathbf{x}))\, d\mathbf{x}$$

for every $\mathbf{F} \in \mathbf{M}^{m\times N}$, and every test map $\mathbf{v}$ in D. This concept does not depend on the domain D, and can, equivalently, be formulated in terms of periodic mappings so that such a density ψ is quasiconvex if

$$\psi(\mathbf{F}) \le \int_D \psi(\mathbf{F} + \nabla \mathbf{v}(\mathbf{y}))\, d\mathbf{y}$$

for all $\mathbf{F} \in \mathbf{M}^{m\times N}$, and every periodic mapping $\mathbf{v} : D \to \mathbf{R}^m$. Now $D \subset \mathbf{R}^N$ is the unit cube.

Unfortunately, the issue is far from settled by simply saying this, since even Morrey realized that it is not at all easy to decide when a given density ψ enjoys this property. For the scalar case, when either of the two dimensions N or m is unity, quasiconvexity reduces to usual convexity. But for genuine vector situations, it is not so. As a matter of fact, necessary and sufficient conditions for quasiconvexity in the vector case ($N, m > 1$) were immediately sought, and important new convexity conditions were introduced:

- Rank-one convexity. A continuous integrand $\psi : \mathbf{M}^{m\times N} \to \mathbf{R}$ is said to be rank-one convex if

$$\psi(t_1\mathbf{F}_1 + t_2\mathbf{F}_2) \le t_1\psi(\mathbf{F}_1) + t_2\psi(\mathbf{F}_2), \quad t_1 + t_2 = 1, t_1, t_2 \ge 0,$$

  whenever the difference $\mathbf{F}_1 - \mathbf{F}_2$ is a rank-one matrix.

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- Polyconvexity. Such an integrand ψ is polyconvex, if it can be rewritten in the form $\psi(\mathbf{F}) = g(\mathbf{M}(\mathbf{F}))$ where $\mathbf{M}(\mathbf{F})$ is the vector of all minors of $\mathbf{F}$, and g is a convex (in the usual sense) function of all its arguments.

One of the main fields where vector variational problems are relevant is non-linear elasticity ([3]). In particular, polyconvexity has played a major role in existence results. See also [8]. A main hypothesis to assume in this area is the rotationally invariance, as well as the behavior for large deformations. See [12] for a discussion on all these notions of convexity under this invariance. Higher-order theories have also been explored, at least from an abstract point of view ([14], [20]). More general concepts of quasiconvexity have been introduced in [16]. Recent interesting results about approximation by polynomials are worth mentioning [17]. Explicit examples of rank-one convex functions can be found in various works: [5], [11], [31], among others. See also [37], [38].

It was very soon recognized that quasiconvexity implies rank-one convexity (by using a special class of test fields), and that polyconvexity is a sufficient condition for quasiconvexity. The task suggested itself as trying to prove or disprove the equivalence of these various kinds of convexity. In the scalar case, all three coincide with usual convexity, so that we are facing a purely vector phenomenon. It turns out that these three notions of convexity are different, and counterexamples of various sorts have been found over the years. See [1], [11], [30], [36].

If we focus on the equivalence of rank-one convexity and quasiconvexity, Morrey conjectured that they are not equivalent ([21]), though later he simply stated it as an unsolved problem ([22]). The issue remained undecided until the surprising counterexamble by V. Sverak ([33]) after some other additional and very interesting results ([31], [32], [34]). What is quite remarkable is that the original counterexample is only valid when $m \geq 3$, and later attempts to extend it for $m = 2$ failed ([4], [27], [29]). Other counterexamples have not been found. References [15], and [19] are also relevant here.

The situation for two-component maps has stayed unsolved, though some evidence in favor of the equivalence has been gathered throughout the years. See [7], [23], [24], [25]. It is also interesting to point out that for quadratic densities, rank-one convexity and quasiconvexity are equivalent regardless of dimensions. This has been known for a long time ([3], [22]), and it is not difficult to prove it by using Plancherel's formula. A different point of view is taken in [5]. Another field where the resolution of this equivalence for two components maps would have an important impact is the theory of quasiconformal maps in the plane. There is a large number of references for this topic. See [2] for a recent account. In particular, if the equivalence between rank-one convexity and quasiconvexity for two component maps turns out to be true, then the norm of the corresponding Beurling-Ahlfors transform equals $p^* - 1$ [18].

In this note, we support that Morrey was right in all cases: for dimensions m and N greater than unity, rank-one convexity does not imply quasiconvexity. We provide an explicit family of maps for $m = N = 2$, parametrized by τ, and argue that, for τ small, they cannot be achieved by lamination. As it is well-known ([28]), this is equivalent to showing that there are rank-one convex functions which are not quasiconvex. The genesis of such family of maps goes back to many efforts of the author in trying to show the equivalence of rank-one convexity and quasiconvexity, up to a point that it was clear the structure of a potential counterexample. A principal issue is how one could rigorously prove that a certain probability measure $\nu \equiv \nu_\tau$, supported in matrices, with a vanishing first moment, and generated by gradients, cannot be achieved by lamination, without finding or constructing a suitable rank-one convex function Ψ for which

$$\langle \Psi, \nu \rangle < \Psi(\mathbf{0}). \tag{1.1}$$

For our candidate $\nu \equiv \nu_\tau$, it may be possible to find one such explicit function Ψ (even in the form of a four-degree polynomial in $\mathbf{M}^{2\times 2}$ as in [33]) for which (1.1) holds, and so such Ψ, though rank-one convex, cannot be quasiconvex. The author has not succeeded in doing so, and so the emphasis is placed in arguing that, for small τ, ν_τ is not, in fact, reachable by lamination.

The structure of the paper is as follows. We immediately introduce that family of gradients parametrized by a real parameter τ, which is bound to be small. Since our strategy focuses in showing that, for sufficiently small τ, those gradients cannot be achieved by lamination, we next recall the definition, and the basic facts about the class of laminates. In Section 4, we revise the case of 3×2-matrices so as to stress how one can conclude that a gradient is not a laminate without finding a rank-one convex which is not quasiconvex. Our discussion for the 3×2 case has been designed to follow as close as possible our strategy for the 2×2 case. For this important 2×2 situation, the proof has to be harder as a specific clear-cut example after the 3×2 counterpart is not possible, but rather one has to rely on a asymptotic situation as a certain parameter τ becomes small. We then concentrate in the main goal of convincing that ν_τ cannot be reached by lamination for small τ.

2. The counterexample

We will consider the following family of piecewise affine maps $\mathbf{u}_\tau : D \subset \mathbf{R}^2 \to \mathbf{R}^2$ parametrized by a real parameter τ. D is the unit cube in $\mathbf{R}^2$. We in fact give their gradients $\nabla \mathbf{u}_\tau : D \to \mathbf{M}^{2\times 2}$ in which matrices

$$
\begin{aligned}
\mathbf{X}_1 &= \begin{pmatrix} 1-\tau & \tau \\ \tau & 1-\tau \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} 1+\tau & 3\tau \\ 3\tau & 1+\tau \end{pmatrix}, \\
\mathbf{X}_3 &= \begin{pmatrix} -1+\tau & 3\tau \\ -\tau & 1+\tau \end{pmatrix}, \quad \mathbf{X}_4 = \begin{pmatrix} -1-\tau & \tau \\ -3\tau & 1-\tau \end{pmatrix}, \\
\mathbf{X}_5 &= \begin{pmatrix} 1+\tau & -\tau \\ 3\tau & -1+\tau \end{pmatrix}, \quad \mathbf{X}_6 = \begin{pmatrix} 1-\tau & -3\tau \\ \tau & -1-\tau \end{pmatrix}, \\
\mathbf{X}_7 &= \begin{pmatrix} -1-\tau & -3\tau \\ -3\tau & -1-\tau \end{pmatrix}, \quad \mathbf{X}_8 = \begin{pmatrix} -1+\tau & -\tau \\ -\tau & -1+\tau \end{pmatrix}.
\end{aligned}
$$

participate according to the geometry provided in Figure 1. Subindices are only shown in this figure. Notice that the underlying barycenter is the zero matrix for all τ.

The appropriate rank-one connections with the appropriate normals occur among them. This is elementary, but worth checking. We need to compute the differences

$$
\begin{aligned}
\mathbf{X}_1 - \mathbf{X}_2 &= \begin{pmatrix} -2\tau & -2\tau \\ -2\tau & -2\tau \end{pmatrix}, \quad \mathbf{X}_1 - \mathbf{X}_4 = \begin{pmatrix} 2 & 0 \\ 4\tau & 0 \end{pmatrix}, \\
\mathbf{X}_1 - \mathbf{X}_6 &= \begin{pmatrix} 0 & 4\tau \\ 0 & 2 \end{pmatrix}, \quad \mathbf{X}_2 - \mathbf{X}_3 = \begin{pmatrix} 2 & 0 \\ 4\tau & 0 \end{pmatrix}, \\
\mathbf{X}_2 - \mathbf{X}_5 &= \begin{pmatrix} 0 & 4\tau \\ 0 & 2 \end{pmatrix}, \quad \mathbf{X}_3 - \mathbf{X}_4 = \begin{pmatrix} 2\tau & 2\tau \\ 2\tau & 2\tau \end{pmatrix}, \\
\mathbf{X}_3 - \mathbf{X}_8 &= \begin{pmatrix} 0 & 4\tau \\ 0 & 2 \end{pmatrix}, \quad \mathbf{X}_4 - \mathbf{X}_7 = \begin{pmatrix} 0 & 4\tau \\ 0 & 2 \end{pmatrix}, \\
\mathbf{X}_5 - \mathbf{X}_6 &= \begin{pmatrix} 2\tau & 2\tau \\ 2\tau & 2\tau \end{pmatrix}, \quad \mathbf{X}_5 - \mathbf{X}_8 = \begin{pmatrix} 2 & 0 \\ 4\tau & 0 \end{pmatrix}, \\
\mathbf{X}_6 - \mathbf{X}_7 &= \begin{pmatrix} 2 & 0 \\ 4\tau & 0 \end{pmatrix}, \quad \mathbf{X}_7 - \mathbf{X}_8 = \begin{pmatrix} -2\tau & -2\tau \\ -2\tau & -2\tau \end{pmatrix}.
\end{aligned}
$$

We therefore see that these differences are rank-one matrices with the appropriate normals according to Figure 1. In addition, we realize that, because only three rank-one matrices occur in those differences, the subspace spanned by those eight matrices is three-dimensional, with basis

$$\begin{pmatrix} 1 & 0 \\ 2\tau & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2\tau \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \tau & \tau \\ \tau & \tau \end{pmatrix}. \tag{2.1}$$

As a matter of fact, the coordinates of the $\mathbf{X}_i$ matrices are easy to compute

$$\mathbf{X}_1 \mapsto (1,1,-1), \quad \mathbf{X}_2 \mapsto (1,1,1), \quad \mathbf{X}_3 \mapsto (-1,1,1), \quad \mathbf{X}_4 \mapsto (-1,1,-1),$$
$$\mathbf{X}_5 \mapsto (1,-1,1), \quad \mathbf{X}_6 \mapsto (1,-1,-1), \quad \mathbf{X}_7 \mapsto (-1,-1,-1), \quad \mathbf{X}_8 \mapsto (-1,-1,1).$$

In this way, if ν_τ designates the underlying (gradient Young) probability measure, then it is supported in the vertices of the unit cube of the subspace $\mathbf{L}_\tau$ generated by (2.1). More specifically

$$\nu_\tau = \frac{3}{16}\left(\delta_{\mathbf{X}_2} + \delta_{\mathbf{X}_4} + \delta_{\mathbf{X}_6} + \delta_{\mathbf{X}_8}\right) \frac{1}{16}\left(\delta_{\mathbf{X}_1} + \delta_{\mathbf{X}_3} + \delta_{\mathbf{X}_5} + \delta_{\mathbf{X}_7}\right),$$

or through the above identification

$$\begin{aligned} \nu_\tau =& \frac{3}{16}\left(\delta_{(1,1,1)} + \delta_{(-1,1,-1)} + \delta_{(1,-1,-1)} + \delta_{(-1,-1,1)}\right) \\ &+ \frac{1}{16}\left(\delta_{(1,1,-1)} + \delta_{(-1,1,1)} + \delta_{(1,-1,1)} + \delta_{(-1,-1,-1)}\right). \end{aligned} \tag{2.2}$$

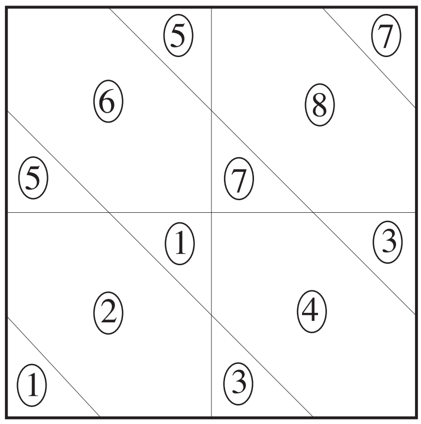

Figure 1. The family of mappings $\mathbf{u}_\tau$.

Note that the representation of ν_τ in (2.2) does not depend on τ, but rank-one directions in $\mathbf{L}_\tau$ do, namely, rank-one directions (x,y,z) must satisfy the equation

$$(1+2\tau)xy + \tau xz + \tau yz = 0 \tag{2.3}$$

if (x,y,z) are coordinates in $\mathbf{L}_\tau$ with respect to the basis in (2.1).

We would like to argue that this gradient Young measure ν_τ cannot be achieved by lamination for τ sufficiently small. See [28]) for a full discussion about gradient Young measures.

3. Laminates

Definition 3.1. *A probability measure ν, supported in $\mathbf{M}^{m\times N}$ with barycenter $\mathbf{0}$, is called a laminate if there are matrices $\mathbf{X}_{i,j}$, and weights $\lambda_{i,j}\geq 0$, for $i\geq 0$, $1\leq j\leq 2^i$ with:*

(1) $\mathbf{X}_{0,1}=\mathbf{0}$.
(2) *Main decomposition step:*
$$(\lambda_{i+1,2j}+\lambda_{i+1,2j-1})\mathbf{X}_{i,j}=\lambda_{i+1,2j}\mathbf{X}_{i+1,2j}+\lambda_{i+1,2j-1}\mathbf{X}_{i+1,2j-1},$$
and every difference
$$\mathbf{Y}_{i,j}\equiv\mathbf{X}_{i+1,2j}-\mathbf{X}_{i+1,2j-1}$$
is a rank-one matrix.
(3) $\lambda_{i,j}=\lambda_{i+1,2j}+\lambda_{i+1,2j-1}$.
(4) *Limit behavior:*
$$\sum_{j=1}^{2^i}\lambda_{i,j}\delta_{\mathbf{X}_{i,j}}\rightharpoonup\nu$$
as $i\to\infty$.

The rank-one condition for each of the differences $\mathbf{Y}_{i,j}$ is the main structural condition. This condition implies that for each such matrix $\mathbf{Y}_{i,j}$ there are vectors $\mathbf{v}_{i,j}\in\mathbf{R}^m$, and $\mathbf{n}_{i,j}\in\mathbf{R}^N$ so that $\mathbf{Y}_{i,j}=\mathbf{v}_{i,j}\otimes\mathbf{n}_{i,j}$. In particular, if we were to go through this decomposition process for each projection of ν onto each of the k-th row of matrices, $k=1,2,\dots,m$, decomposition directions for all components would simultaneously correspond to $\mathbf{n}_{i,j}$. This identical decomposition directions for all rows is what ties together the (vector) measure ν. The common direction $\mathbf{n}_{i,j}$ also has a very clear geometrical interpretation as the "normal to the layers" when a laminate is understood as a gradient Young measure (see [28]).

Definition 3.2. *A real function $\Psi:\mathbf{M}^{m\times N}\to\mathbf{R}$ is rank-one convex if*
$$\Psi(t\mathbf{Z}_1+(1-t)\mathbf{Z}_0)\leq t\Psi(\mathbf{Z}_1)+(1-t)\Psi(\mathbf{Z}_0)$$
whenever the difference matrix $\mathbf{Z}_1-\mathbf{Z}_0$ is of rank one.

With these two basic definitions, it is elementary to show Jensen's inequality.

Lemma 3.1. *Let ν be a laminate as above, and Ψ, a rank-one convex function. Then*
$$\langle\Psi,\nu\rangle\geq\Psi(\mathbf{0}).\tag{3.1}$$

What is interesting is that this inequality is a characterization of laminates ([26], [28]).

Theorem 3.2. *Let ν be a probability measure as before. Then ν is a laminate in the sense of Definition (3.1) if and only if for every rank-one convex function Ψ, (3.1) is valid.*

This result suggests a potential strategy to prove that there are rank-one convex functions which are not quasiconvex, if we can show that a certain probability measure ν, underlying a gradient $\nabla\mathbf{u}$ as in Section 2, is not a laminate, for then, according to Theorem 3.2, there must be at least one rank-one convex function $\overline{\Psi}$ for which (3.1) cannot be correct. This function $\overline{\Psi}$ cannot be quasiconvex because
$$\int_Q\overline{\Psi}(\nabla\mathbf{u}(\mathbf{x}))\,d\mathbf{x}=\langle\overline{\Psi},\nu\rangle<\overline{\Psi}(\mathbf{0}).$$
This pretends to be our strategy.

Since we will be working with the full family of gradients parametrized by τ introduced in Section 2, it is important to write down explicitly such situation. Assume ν_τ is a laminate for all positive τ. Then there are matrices $\mathbf{X}^{(\tau)}_{i,j}$, and weights $\lambda^{(\tau)}_{i,j}\geq 0$, for $i\geq 0$, $1\leq j\leq 2^i$ with:

(1) $\mathbf{X}^{(\tau)}_{0,1}=\mathbf{0}$, for all τ.
(2) Main decomposition step:

$$(\lambda^{(\tau)}_{i+1,2j}+\lambda^{(\tau)}_{i+1,2j-1})\mathbf{X}^{(\tau)}_{i,j}=\lambda^{(\tau)}_{i+1,2j}\mathbf{X}^{(\tau)}_{i+1,2j}+\lambda^{(\tau)}_{i+1,2j-1}\mathbf{X}^{(\tau)}_{i+1,2j-1},$$

and every difference

$$\mathbf{Y}^{(\tau)}_{i,j}\equiv\mathbf{X}^{(\tau)}_{i+1,2j}-\mathbf{X}^{(\tau)}_{i+1,2j-1}$$

is a rank-one matrix.
(3) $\lambda^{(\tau)}_{i,j}=\lambda^{(\tau)}_{i+1,2j}+\lambda^{(\tau)}_{i+1,2j-1}$.
(4) Limit behavior:

$$\sum_{j=1}^{2^i}\lambda^{(\tau)}_{i,j}\delta_{\mathbf{X}^{(\tau)}_{i,j}}\rightharpoonup\nu_\tau$$

as $i\to\infty$, for all $\tau>0$, or else

$$\nu_\tau=\sum_{j=1}^{2^i}\lambda^{(\tau)}_{i,j}\delta_{\mathbf{X}^{(\tau)}_{i,j}}\tag{3.2}$$

for some finite i, that may depend on τ.

We also know that all matrices $\mathbf{X}^{(\tau)}_{i,j}$ participating in these decompositions should belong to the convex hull of the eight matrices in the support of ν_τ.

Since our main argument will be asymptotic as $\tau\searrow 0$, the following concept will play a relevant role. It identifies the rank-one directions we cannot be dispensed with in the limit.

Definition 3.3. *With the notation just introduced, we say that a certain unit, rank-one matrix* $\mathbf{Y}$ *is a proper decomposition direction for the family of laminates* $\{\nu_\tau\}$ *as* $\tau\searrow 0$*, if there is* $i\geq 0$*,* $1\leq j\leq 2^i$*, and a subsequence* $\tau_k\searrow 0$ *such that*

$$\frac{\mathbf{Y}^{(\tau_k)}_{i,j}}{|\mathbf{Y}^{(\tau_k)}_{i,j}|}\to\mathbf{Y},\quad \frac{\lambda^{(\tau_k)}_{i+1,2j}}{\lambda^{(\tau_k)}_{i+1,2j}+\lambda^{(\tau_k)}_{i+1,2j-1}}\to\overline{\lambda}_{i,j}\in(0,1),\qquad \textit{as } k\to\infty.$$

The emphasis is placed in the fact that $0<\overline{\lambda}_{i,j}<1$.

4. The 3×2 situation

As a preliminary step to better understand the main argument behind a proof to check that a certain probability measure is not reachable by lamination, we are going to treat, from this viewpoint, the case of 3×2 matrices. This material is taken from [27].

Consider the piecewise-affine map $\mathbf{u} : D \subset \mathbf{R}^2 \to \mathbf{R}^3$ determined by its gradient $\nabla\mathbf{u} : D \to \mathbf{M}^{3\times 2}$ in which matrices

$$\mathbf{X}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix},$$
$$\mathbf{X}_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{X}_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix},$$
$$\mathbf{X}_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{X}_6 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -1 & -1 \end{pmatrix},$$
$$\mathbf{X}_7 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{pmatrix}, \quad \mathbf{X}_8 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

participate according again to the geometry in Figure 1. The underlying barycenter is the zero matrix $\mathbf{0}$. Once again, it is elementary to check the various rank-one relations among those matrices with the corresponding normals, so that the underlying (homogeneous, gradient Young) probability measure is

$$\nu = \frac{3}{16}\left(\delta_{\mathbf{X}_2} + \delta_{\mathbf{X}_4} + \delta_{\mathbf{X}_6} + \delta_{\mathbf{X}_8}\right)\frac{1}{16}\left(\delta_{\mathbf{X}_1} + \delta_{\mathbf{X}_3} + \delta_{\mathbf{X}_5} + \delta_{\mathbf{X}_7}\right).$$

Specifically, those differences are

$$\mathbf{X}_1 - \mathbf{X}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -2 & -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \end{pmatrix},$$
$$\mathbf{X}_1 - \mathbf{X}_4 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix},$$
$$\mathbf{X}_1 - \mathbf{X}_6 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix},$$

and then

$$\mathbf{X}_2 - \mathbf{X}_3 = \mathbf{X}_1 - \mathbf{X}_4, \quad \mathbf{X}_2 - \mathbf{X}_5 = \mathbf{X}_1 - \mathbf{X}_6, \quad \mathbf{X}_3 - \mathbf{X}_4 = \mathbf{X}_2 - \mathbf{X}_1,$$
$$\mathbf{X}_3 - \mathbf{X}_8 = \mathbf{X}_1 - \mathbf{X}_6, \quad \mathbf{X}_4 - \mathbf{X}_7 = \mathbf{X}_1 - \mathbf{X}_6, \quad \mathbf{X}_5 - \mathbf{X}_6 = \mathbf{X}_1 - \mathbf{X}_2,$$
$$\mathbf{X}_6 - \mathbf{X}_7 = \mathbf{X}_1 - \mathbf{X}_4, \quad \mathbf{X}_7 - \mathbf{X}_8 = \mathbf{X}_1 - \mathbf{X}_2.$$

It is again easy to check that all those eight matrices belong to the subspace $\mathbf{L}$ spanned by the basis

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}, \tag{4.1}$$

and that ν can be identified, using coordinates with respect to this basis, with

$$\begin{aligned} \nu =& \frac{3}{16}\left(\delta_{(1,1,1)} + \delta_{(-1,1,-1)} + \delta_{(1,-1,-1)} + \delta_{(-1,-1,1)}\right) \\ &+ \frac{1}{16}\left(\delta_{(1,1,-1)} + \delta_{(-1,1,1)} + \delta_{(1,-1,1)} + \delta_{(-1,-1,-1)}\right). \end{aligned} \tag{4.2}$$

Matrices in $\mathbf{L}$ are represented in the form

$$\begin{pmatrix} x & 0 \\ 0 & y \\ z & z \end{pmatrix},$$

and so rank-one directions in $\mathbf{L}$ correspond exactly to the three basis matrices with coordinates $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. This means that in building laminates supported in $\mathbf{L}$, we are only entitled to use these three decomposition directions. It is then clear that the asymmetry of weights in (4.2) cannot be achieved by using just these three directions in $\mathbf{L}$ in the process determining a laminate (Definition 3.1). In fact, the only laminate supported in those eight matrices with barycenter $\mathbf{0}$ is the one with symmetric weights

$$\begin{aligned}\nu =& \frac{1}{8}\left(\delta_{(1,1,1)} + \delta_{(-1,1,-1)} + \delta_{(1,-1,-1)} + \delta_{(-1,-1,1)}\right) \\ &+ \frac{1}{8}\left(\delta_{(1,1,-1)} + \delta_{(-1,1,1)} + \delta_{(1,-1,1)} + \delta_{(-1,-1,-1)}\right).\end{aligned}$$

So, ν in (4.2) cannot be a laminate. See Figure 2.

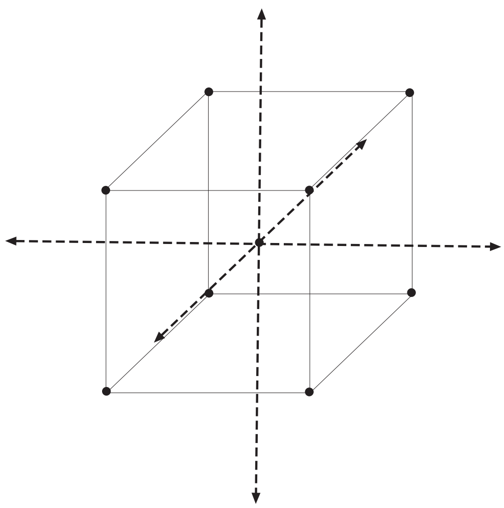

Figure 2. The 3×2 situation.

5. The 2×2 case

We go back to the problem which is the main concern of this contribution. Notice how in this case, a subspace $\mathbf{L}$ like the one in the preceding section cannot be found. This is essentially what makes the search for a counterexample in the 2×2 situation much more involved.

5.1. **The limit measure.** Because the probability measure ν_τ in (2.2), associated with $\nabla \mathbf{u}_\tau$, is independent of τ, we can take advantage of this feature to examine the (singular) limit as $\tau \searrow 0$. Indeed, the limit measure will be the same

$$\begin{aligned}\nu_0 =& \frac{1}{16}(3\delta_{(1,1,1)} + 3\delta_{(1,-1,-1)} + 3\delta_{(-1,1,-1)} + 3\delta_{(-1,-1,1)} \\ &+ \delta_{(-1,1,1)} + \delta_{(1,-1,1)} + \delta_{(1,1,-1)} + \delta_{(-1,-1,-1)}),\end{aligned}$$

though the cone Λ_0 of rank-one directions will degenerate to the condition $xy = 0$ which is the limit of the equation (2.3) as $\tau \searrow 0$. ν_0 should be a laminate with respect to this cone of directions. It is elementary to check that this is indeed so:

$$(0,0,0) = \frac{1}{2}(0,-1,0) + \frac{1}{2}(0,1,0), \quad (0,1,0) - (0,-1,0) = (0,2,0) \in \Lambda_0,$$
$$(0,-1,0) = \frac{1}{2}(1,-1,-1/2) + \frac{1}{2}(-1,-1,1/2), \quad (1,-1,-1/2) - (-1,-1,1/2) = (2,0,-1) \in \Lambda_0,$$
$$(0,1,0) = \frac{1}{2}(1,1,1/2) + \frac{1}{2}(-1,1,-1/2), \quad (1,1,1/2) - (-1,1,-1/2) = (2,0,1) \in \Lambda_0,$$
$$(1,-1,-1/2) = \frac{1}{4}(1,-1,1) + \frac{3}{4}(1,-1,-1), \quad (1,-1,1) - (1,-1,-1) = (0,0,2) \in \Lambda_0,$$
$$(-1,-1,1/2) = \frac{1}{4}(-1,-1,-1) + \frac{3}{4}(-1,-1,1), \quad (-1,-1,-1) - (-1,-1,1) = (0,0,-2) \in \Lambda_0,$$
$$(1,1,1/2) = \frac{1}{4}(1,1,-1) + \frac{3}{4}(1,1,1), \quad (1,1,-1) - (1,1,1) = (0,0,-2) \in \Lambda_0,$$
$$(-1,1,-1/2) = \frac{1}{4}(-1,1,1) + \frac{3}{4}(-1,1,-1), \quad (-1,1,1) - (-1,1,-1) = (0,0,2) \in \Lambda_0.$$

There is another possibility that can easily found by replacing the decomposition directions $(2,0,-1)$ and $(2,0,1)$, by $(1,0,-2)$ and $(1,0,2)$, respectively, and then using the direction $(1,0,0)$ instead of $(0,0,1)$. Figure 3 can help in visualizing the situation. But there are some other possibilities.

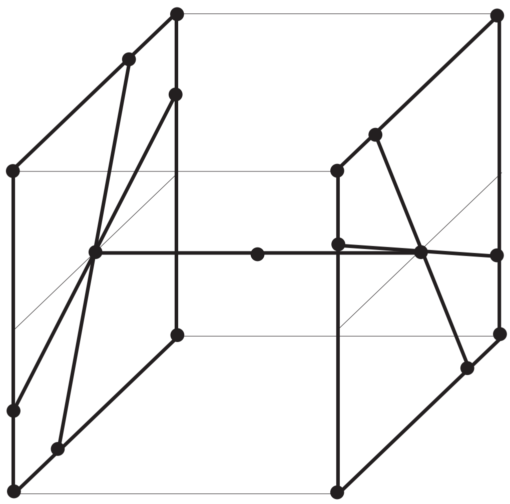

Figure 3. The limit situation.

Lemma 5.1. *There is a constant $c_0 > 0$, such that proper decomposition directions to achieve ν_0 through lamination can only be of the form $(c,0,1)$ or $(0,c,1)$ with either $c = 0$, $c = \infty$, or else $c_0 \le |c| \le 1/c_0$.*

Proof. Recall that Λ_0 is the cone of directions (x,y,z) with $xy = 0$. The statement is elementary once we realize that only decomposition directions (x,y,z) with $x = 0$ or $y = 0$ can be used,

and that the partial barycenter for the four mass points corresponding to either $y=-1$, $y=1$, $x=-1$, or $x=1$ is in all cases the center of the corresponding square. The asymetric distribution of weights on each of these four faces of the cube takes place along the directions $(1,0,1)$, $(-1,0,1)$, $(0,-1,1)$, and $(0,1,1)$, respectively. This asymetry forbids decomposition directions to be arbitrarily close to $(0,0,1)$, to $(1,0,1)$, or to $(0,1,0)$, which are the directions corresponding to $c=0$, and to $c=\infty$. □

Let Λ_τ be the set of rank-one directions in $\mathbf{L}_\tau$ complying with equation (2.3). The feasible directions in the lemma do not belong to Λ_τ, though, by continuity, they are close to belonging to it. The directions

$$\left(c, \frac{-\tau c}{c+\tau(1+2c)}, 1\right), \quad \left(\frac{-\tau c}{c+\tau(1+2c)}, c, 1\right), \tag{5.1}$$

do belong to Λ_τ. As already pointed out above, for $c=0$, we have $(0,0,1)$, while for $c=\infty$, the directions are $(1,0,0)$, and $(0,1,0)$.

5.2. **The contradiction.** Suppose the probability measure ν_τ in (2.2) can be reached by lamination for all small τ. There might be many decomposition directions involved in the process for the laminate, some of which may even depend upon τ, but, by continuity, only those which are close to the ones used for the limit case $\tau=0$ in the preceding subsection, can be proper according to Definition 3.3. By this we mean that corresponding relative weights stay away from zero and one, as $\tau\searrow 0$. Those proper decomposition directions are

$$(1,0,0), \quad (0,1,0), \quad (0,0,1),$$

and those in (5.1) for $c_0 \le |c| \le 1/c_0$. The true proper, decomposition directions for small τ might not be exactly these ones, but their differences ought to converge to zero with τ. Let us examine those decomposition directions, and find the real rank-one matrices underlying those directions through the identification associated with the basis in (2.1)

$$(x,y,z)\mapsto \begin{pmatrix} x+\tau z & 2\tau y+\tau z\\ 2\tau x+\tau z & y+\tau z\end{pmatrix}.$$

Indeed, we have

$$(1,0,0)\mapsto \begin{pmatrix} 1 & 0\\ 2\tau & 0\end{pmatrix}\mapsto (1,0), \quad (0,1,0)\mapsto \begin{pmatrix} 0 & 2\tau\\ 0 & 1\end{pmatrix}\mapsto (0,1), \quad (0,0,1)\mapsto \begin{pmatrix} \tau & \tau\\ \tau & \tau\end{pmatrix}\mapsto (1,1),$$

for either $c=0$, or $c=\infty$, while for finite c, vectors and matrices are

$$\left(c, \frac{-\tau c}{c+\tau(1+2c)}, 1\right)\mapsto \begin{pmatrix} c+\tau & \tau(c+\tau)/(c+\tau(1+2c))\\ \tau(1+2c) & \tau^2(1+2c)/(c+\tau(1+2c))\end{pmatrix}\mapsto (1, \tau/(c+\tau(1+2c))),$$
$$\left(\frac{-\tau c}{c+\tau(1+2c)}, c, 1\right)\mapsto \begin{pmatrix} \tau(c+\tau)/(c+\tau(1+2c)) & c+\tau\\ \tau^2(1+2c)/(c+\tau(1+2c)) & \tau(1+2c)\end{pmatrix}\mapsto (\tau/(c+\tau(1+2c)), 1).$$

The final vectors written for each case correspond to the true lamination directions, i.e. the normal to the parallel layers. Hence, we conclude that the proper normals used in the lamination process for ν_τ have to be close to one of the three directions $(1,0)$, $(0,1)$, or $(1,1)$. The contradiction arises because it is impossible to get the asymmetry in the weights for ν_τ by using these three decomposition directions simultaneously in both components, much in the same way as with the 3×2 situation.

To argue further on this issue, let us focus on the probability measures $\nu_\tau^{(l)}$, $l=1,2$, corresponding to the two components in (2.2). Namely,

$$\nu_\tau^{(l)} = \frac{3}{16}\left(\delta^{(l)}_{\mathbf{X}_2}+\delta^{(l)}_{\mathbf{X}_4}+\delta^{(l)}_{\mathbf{X}_6}+\delta^{(l)}_{\mathbf{X}_8}\right)\frac{1}{16}\left(\delta^{(l)}_{\mathbf{X}_1}+\delta^{(l)}_{\mathbf{X}_3}+\delta^{(l)}_{\mathbf{X}_5}+\delta^{(l)}_{\mathbf{X}_7}\right), \text{ for } l=1,2. \tag{5.2}$$

Here $\mathbf{X}_s^{(l)}$ is the l-th row of $\mathbf{X}_s$:

$$\mathbf{X}_1^{(1)} = \begin{pmatrix} 1-\tau & \tau \end{pmatrix}, \mathbf{X}_1^{(2)} = \begin{pmatrix} \tau & 1-\tau \end{pmatrix}, \quad \mathbf{X}_2^{(1)} = \begin{pmatrix} 1+\tau & 3\tau \end{pmatrix}, \mathbf{X}_2^{(2)} = \begin{pmatrix} 3\tau & 1+\tau \end{pmatrix},$$

$$\mathbf{X}_3^{(1)} = \begin{pmatrix} -1+\tau & 3\tau \end{pmatrix}, \mathbf{X}_3^{(2)} = \begin{pmatrix} -\tau & 1+\tau \end{pmatrix}, \quad \mathbf{X}_4^{(1)} = \begin{pmatrix} -1-\tau & \tau \end{pmatrix}, \mathbf{X}_4^{(2)} = \begin{pmatrix} -3\tau & 1-\tau \end{pmatrix},$$

$$\mathbf{X}_5^{(1)} = \begin{pmatrix} 1+\tau & -\tau \end{pmatrix}, \mathbf{X}_5^{(2)} = \begin{pmatrix} 3\tau & -1+\tau \end{pmatrix}, \quad \mathbf{X}_6^{(1)} = \begin{pmatrix} 1-\tau & -3\tau \end{pmatrix}, \mathbf{X}_6^{(2)} = \begin{pmatrix} \tau & -1-\tau \end{pmatrix},$$

$$\mathbf{X}_7^{(1)} = \begin{pmatrix} -1-\tau & -3\tau \end{pmatrix}, \mathbf{X}_7^{(2)} = \begin{pmatrix} -3\tau & -1-\tau \end{pmatrix}, \quad \mathbf{X}_8^{(1)} = \begin{pmatrix} -1+\tau & -\tau \end{pmatrix}, \mathbf{X}_8^{(2)} = \begin{pmatrix} -\tau & -1+\tau \end{pmatrix}.$$

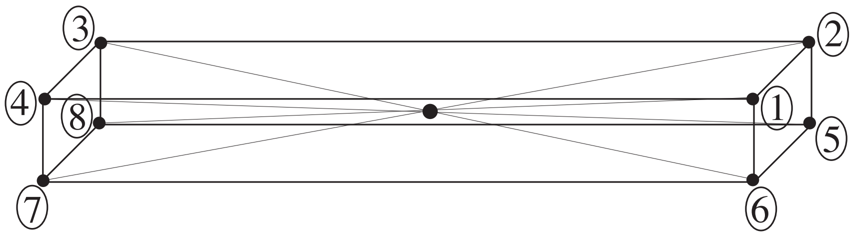

Figure 4. The first component of $\mathbf{u}_\tau$.

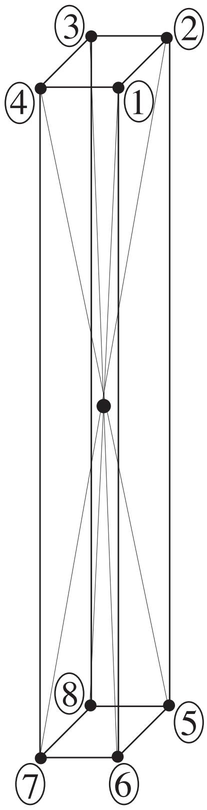

Figure 5. The second component of $\mathbf{u}_\tau$.

The eight vectors participating in $\nu_\tau^{(1)}$ are

$$\mathbf{X}_1^{(1)} = \begin{pmatrix}1-\tau & \tau\end{pmatrix}, \quad \mathbf{X}_2^{(1)} = \begin{pmatrix}1+\tau & 3\tau\end{pmatrix},$$
$$\mathbf{X}_3^{(1)} = \begin{pmatrix}-1+\tau & 3\tau\end{pmatrix}, \quad \mathbf{X}_4^{(1)} = \begin{pmatrix}-1-\tau & \tau\end{pmatrix},$$
$$\mathbf{X}_5^{(1)} = \begin{pmatrix}1+\tau & -\tau\end{pmatrix}, \quad \mathbf{X}_6^{(1)} = \begin{pmatrix}1-\tau & -3\tau\end{pmatrix},$$
$$\mathbf{X}_7^{(1)} = \begin{pmatrix}-1-\tau & -3\tau\end{pmatrix}, \quad \mathbf{X}_8^{(1)} = \begin{pmatrix}-1+\tau & -\tau\end{pmatrix},$$

while those occurring in $\nu_\tau^{(2)}$ are

$$\mathbf{X}_1^{(2)} = \begin{pmatrix}\tau & 1-\tau\end{pmatrix}, \quad \mathbf{X}_2^{(2)} = \begin{pmatrix}3\tau & 1+\tau\end{pmatrix},$$
$$\mathbf{X}_3^{(2)} = \begin{pmatrix}-\tau & 1+\tau\end{pmatrix}, \quad \mathbf{X}_4^{(2)} = \begin{pmatrix}-3\tau & 1-\tau\end{pmatrix},$$
$$\mathbf{X}_5^{(2)} = \begin{pmatrix}3\tau & -1+\tau\end{pmatrix}, \quad \mathbf{X}_6^{(2)} = \begin{pmatrix}\tau & -1-\tau\end{pmatrix},$$
$$\mathbf{X}_7^{(2)} = \begin{pmatrix}-3\tau & -1-\tau\end{pmatrix}, \quad \mathbf{X}_8^{(2)} = \begin{pmatrix}-\tau & -1+\tau\end{pmatrix}.$$

It is easy to represent both probability measures in the plane. See Figures 4 and 5.

Since τ tends to zero, the proper decomposition directions are, as indicated above, $(1,0)$, $(0,1)$, $(1,1)$. When we use a decomposition rank-one matrix in ν_τ, associated with a direction close to $(1,0)$ in both components, that direction might help in creating the asymmetry of weights for $\nu_\tau^{(1)}$ in (5.2) (see Figure 6), but not for $\nu_\tau^{(2)}$, precisely because the support of $\nu_\tau^{(1)}$ is stretched out in that direction, but the support of $\nu_\tau^{(2)}$ is arbitrarily thin in that same direction. Since the asymmetry of weights has to be created simultaneously in both components, it is impossible for a direction close to $(1,0)$ to do so. Exactly the same argument, used symmetrically in both components, shows that a decomposition direction close to $(0,1)$ cannot produce the asymmetry of weights in ν_τ. But under these circumstances, neither a direction close to $(1,1)$ by itself can produce that asymmetry. Only can the probability measure

$$\overline{\nu}_\tau = \frac{1}{8}\left(\delta_{\mathbf{X}_2} + \delta_{\mathbf{X}_4} + \delta_{\mathbf{X}_6} + \delta_{\mathbf{X}_8} + \delta_{\mathbf{X}_1} + \delta_{\mathbf{X}_3} + \delta_{\mathbf{X}_5} + \delta_{\mathbf{X}_7}\right),$$

be a laminate for small τ.

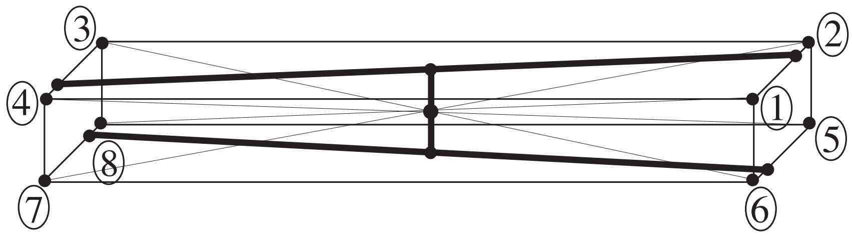

Figure 6. Assymetry of weights with the $(1,0)$ direction.

6. A final remark

It is interesting to note that if we are allowed to perturbed a bit some of the matrices $\mathbf{X}'$s occurring in ν_τ, then we would have a laminate. Indeed, taking advantage of the decompositions for the limit rank-one cone Λ_0 written earlier, and using the corresponding approximating directions from the cone Λ_τ, one can redo essentially the same decompositions replacing the directions

$(2,0,1)$ and $(2,0,-1)$ by the two first ones in (5.1), respectively, and adjusting a little bit the weights. See Figure 7.

Specifically, one arrives at

$$
\begin{aligned}
(0,0,0) &= a_1(\tau)(0,-1-\frac{\tau}{2+3\tau},0)+a_2(\tau)(0,1+\frac{\tau}{2+5\tau},0),\\
(0,1+\frac{\tau}{2+5\tau},0)-(0,-1-\frac{\tau}{2+3\tau},0) &= (0,2+4\tau\frac{1+2\tau}{(2+3\tau)(2+5\tau)},0)\in\Lambda_\tau,\\
a_1(\tau)=\frac{(2+6\tau)(2+3\tau)}{8+36\tau+38\tau^2}, &\quad a_2(\tau)=\frac{(2+4\tau)(2+5\tau)}{8+36\tau+38\tau^2},\\
(0,-1-\frac{\tau}{2+3\tau},0) &= \frac{1}{2}(1,-1,-1/2)+\frac{1}{2}(-1,-1-\frac{2\tau}{2+3\tau},1/2),\\
(1,-1,-1/2)-(-1,-1-\frac{2\tau}{2+3\tau},1/2) &= (2,\frac{2\tau}{2+3\tau},-1)\in\Lambda_\tau,\\
(0,1+\frac{\tau}{2+5\tau},0) &= \frac{1}{2}(1,1,1/2)+\frac{1}{2}(-1,1+\frac{2\tau}{2+5\tau},-1/2),\\
(1,1,1/2)-(-1,1+\frac{2\tau}{2+5\tau},-1/2) &= (2,\frac{-2\tau}{2+5\tau},1)\in\Lambda_\tau,\\
(1,-1,-1/2)=\frac{1}{4}(1,-1,1)+\frac{3}{4}(1,-1,-1), &\quad (1,-1,1)-(1,-1,-1)=(0,0,2)\in\Lambda_\tau,\\
(-1,-1-\frac{2\tau}{2+3\tau},1/2) &= \frac{1}{4}(-1,-1-\frac{2\tau}{2+3\tau},-1)+\frac{3}{4}(-1,-1-\frac{2\tau}{2+3\tau},1),\\
(-1,-1-\frac{2\tau}{2+3\tau},-1)-(-1,-1-\frac{2\tau}{2+3\tau},1) &= (0,0,-2)\in\Lambda_\tau,\\
(1,1,1/2)=\frac{1}{4}(1,1,-1)+\frac{3}{4}(1,1,1), &\quad (1,1,-1)-(1,1,1)=(0,0,-2)\in\Lambda_\tau,\\
(-1,1+\frac{2\tau}{2+5\tau},-1/2) &= \frac{1}{4}(-1,1+\frac{2\tau}{2+5\tau},1)+\frac{3}{4}(-1,1+\frac{2\tau}{2+5\tau},-1),\\
(-1,1+\frac{2\tau}{2+5\tau},1)-(-1,1+\frac{2\tau}{2+5\tau},-1) &= (0,0,2)\in\Lambda_\tau.
\end{aligned}
$$

The laminate that stems from this decomposition process is

$$
\begin{aligned}
\mu_\tau = &\frac{3a_1(\tau)}{8}\delta_{(1,-1,-1)}+\frac{a_1(\tau)}{8}\delta_{(1,-1,1)}+\frac{3a_2(\tau)}{8}\delta_{(1,1,1)}+\frac{a_2(\tau)}{8}\delta_{(1,1,-1)}\\
&+\frac{3a_1(\tau)}{8}\delta_{(-1,-1-\frac{2\tau}{2+3\tau},1)}+\frac{a_1(\tau)}{8}\delta_{(-1,-1-\frac{2\tau}{2+3\tau},-1)}\\
&+\frac{3a_2(\tau)}{8}\delta_{(-1,1+\frac{2\tau}{2+5\tau},-1)}+\frac{a_2(\tau)}{8}\delta_{(-1,1+\frac{2\tau}{2+5\tau},1)}.
\end{aligned}
$$

Notice that

$$a_1(\tau),a_2(\tau)\to 1/2,\quad a_2'(\tau),a_2'(\tau)\to 0$$

as $\tau\to 0$. It is the fact that mass points in ν_τ are fixed, independent of τ, and symmetrically located that prevents ν_τ from being a laminate.

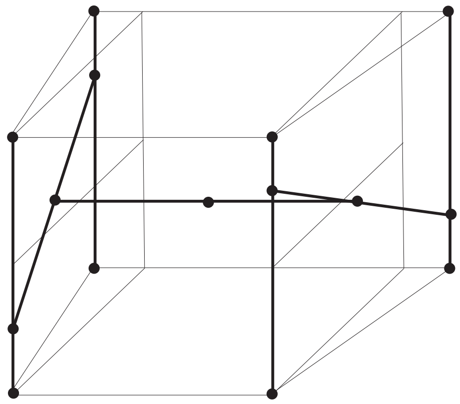

Figure 7. The τ-laminate.

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